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An efficient algorithm for solving multi-pantograph equation systems

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ABSTRACT

In this paper, we present a numerical approach for solving the system of multi-pantograph equations with mixed conditions. This system is usually difficult to solve analytically. By expanding the approximate solutions by means of the Bessel functions of first kind with unknown coefficients, the proposed approach consists of reducing the problem to a linear algebraic equation system. The unknown coefficients of the Bessel functions of first kind are computed using the matrix operations of derivatives together with the collocation method. An error estimation is given. The reliability and efficiency of the proposed scheme are demonstrated by some numerical examples. All of the numerical computations have been performed on a computer with the aid of a program written in Matlab.

1. Introduction

In recent years, many authors have studied numerical methods such as the variational iteration method [1], θ-methods [2], the Taylor matrix method [3], the reproducing kernel space method [4], the Adomian decomposition method [5] for approximate solutions of the multi-pantograph equation

$$y'(t) = \lambda y(t) + \sum_{j=1}^{J} \mu_j(t) y(q_j t) + g(t).$$

Additionally, the approximate solutions of generalized pantograph equations have been obtained by using the homotopy method [6], the Taylor polynomial approach [7], the variational iteration method [8], the Bessel collocation method [9], the Taylor method [10,11]. Also, Brunner et al. [12] have used the Galerkin methods for solutions of delayed differential equations of Pantograph type.

Recently, Yüzbaşı et al. [9,13–17] have studied the Bessel matrix and collocation methods for numerical solutions of the neutral delay differential equations, the pantograph equations, the Lane–Emden differential equations, Fredholm integro-differential equations and Volrerra integral and Fredholm integro-differential equation systems.

In this study, we will develop the matrix and collocation methods studied in [9,13–16] for the approximate solutions of the system of multi-pantograph equations

$$\sum_{j=1}^{k} \beta_{i,j}(t) y_j^{(1)}(t) = \sum_{j=1}^{k} \gamma_{i,j}(t) y_j(t) + \sum_{r=1}^{R} \sum_{j=1}^{k} \mu_{i,j}^{r}(t) y_j(q_r t) + g_i(t), \quad i = 1, 2, \ldots, k, \ 0 \le a \le t \le b \tag{1}$$

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with the mixed conditions

$$\sum_{n=1}^{k}(\phi_{n,i}y_n(a)+\psi_{n,i}y_n(b))=\lambda_i,\quad n=1,2,\ldots,k, \tag{2}$$

where $y_j(t),\ (i=1,2,\ldots,k)$ are the unknown functions, $\beta_{i,j}(t),\ \gamma_{i,j}(t),\ \mu^r_{i,j}(t)$ and $g_i(t)$ are the functions defined on interval $a\le t\le b$, and also q_r, $\phi_{n,i},\ \psi_{n,i}$ and λ_i are appropriate constants.

Our purpose is to obtain the approximate solutions of system (1) expressed in the truncated Bessel series form

$$y_i(t)=\sum_{n=0}^{N}a_{i,n}J_n(t),\quad i=1,2,\ldots,k \tag{3}$$

so that $a_{i,n},\ n=0,1,2,\ldots,N$ are the unknown Bessel coefficients, N is any chosen positive integer such that $N\ge 1$, and $J_n(t),\ n=0,1,2,\ldots,N$ are the Bessel functions of the first kind defined by

$$J_n(t)=\sum_{k=0}^{\left[\!\left[\frac{N-n}{2}\right]\!\right]}\frac{(-1)^k}{k!(k+n)!}\left(\frac{t}{2}\right)^{2k+n},\quad n\in\mathbb{N},\ 0\le t<\infty.$$

This paper is arranged as follows:

We give some properties of the Bessel functions in Section 2. In Section 3, we introduce the fundamental matrix relations to find the matrix forms of each term of the system (1). The method for gaining approximate solutions is described in Section 4. In Section 5, we present an error estimation for the Bessel polynomial solutions. We illustrate some numerical examples to clarify the method in Section 6. Section 7 presents a brief summary of this article.

2. Some properties of the Bessel functions

The Bessel differential equation is the linear second-order ordinary differential equation given by

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+(t^2-p^2)y=0$$

so that p is a non-negative real number. This equation is solved using series solutions. The general solution of this equation has the form

$$y=C_1J_p(t)+C_2Y_p(t)$$

where C_1 and C_2 are constants,

$$J_p(t)=\sum_{k=0}^{\left[\!\left[\frac{N-p}{2}\right]\!\right]}\frac{(-1)^k}{k!(k+p)!}\left(\frac{t}{2}\right)^{2k+p}$$

and

$$Y_p(t)=\frac{2}{\pi}\left\{\left(\ln\left(\frac{t}{2}\right)+\gamma\right)J_p(t)-\frac{1}{2}\sum_{n=0}^{p-1}\frac{(p-n-1)!}{n!}\left(\frac{t}{2}\right)^{2n-p}\right.$$
$$\left.+\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\sum_{k=1}^{n}\frac{1}{k}\sum_{k=1}^{n+p}\frac{1}{k}\right)\left[\frac{1}{n!(n+p)!}\left(\frac{t}{2}\right)^{2n+p}\right]\right\}.$$

Here, $\gamma\cong 0.5772156$ is Euler's constant and $J_p(t)$ and $Y_p(t)$ are called the Bessel functions of the first kind and the Bessel functions of the second kind [18], respectively.

The orthogonality relation [19] over the interval $[0,b]$ with respect to weight function $w(\rho)=\rho$ is given by

$$\int_0^b J_n\left(\frac{v_{nm}}{b}\rho\right)J_n\left(\frac{v_{nk}}{b}\rho\right)\rho d\rho=\begin{cases}0, & m=k\\ \frac{b^2}{2}[J_{n+1}(v_{nm})]^2, & m\ne k\end{cases}$$

where $\rho\in[0,b]$ and v_{nm} is the mth root of the Bessel function $J_n(t)=0$, i.e. $J_n(v_{nm})=0$.

The orthogonality relation is used in determining the coefficients in an expansion of a function in terms of a series of Bessel functions.

If the function $f(t)$ is to be expanded in the range $0 < t < a$, we write:

$$f(t) = \sum_{n=1}^{\infty} c_n J_m(k_n t),$$

where the k_n are chosen so that $J_m(k_n a) = 0$. The coefficients in the expansion [19] are then given by:

$$c_n = \frac{\int_0^a f(t) J_m(k_n t) t dt}{\frac{a^2}{2} J_{m+1}^2(k_n a)}.$$

For integer order $p = n, J_n$ it is often defined via a Laurent series for a generating function [18]

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

, an approach used by Hansen in 1843.

3. Required matrix relations for the Bessel polynomial approximation

Firstly, let us write the Bessel functions of first kind $J_n(t)$ in the matrix form

$$\mathbf{J}^T(t) = \mathbf{D}\mathbf{T}^T(t) \Leftrightarrow \mathbf{J}(t) = \mathbf{T}(t)\mathbf{D}^T, \tag{4}$$

where

$$\mathbf{J}(t) = \begin{bmatrix} J_0(t) & J_1(t) & \cdots & J_N(t) \end{bmatrix}, \qquad \mathbf{T}(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^N \end{bmatrix} \quad \text{and}$$

if N is odd,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N-1}{2}\right)!2^{N-1}} & 0 \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & 0 & \frac{(-1)^{\frac{N-1}{2}}}{\left(\frac{N-1}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N}} \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & \frac{(-1)^{\frac{N-3}{2}}}{\left(\frac{N-3}{2}\right)!\left(\frac{N+1}{2}\right)!2^{N-1}} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)},$$

if N is even,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{0!0!2^0} & 0 & \frac{-1}{1!1!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N}{2}}}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!2^{N}} \\ 0 & \frac{1}{0!1!2^1} & 0 & \cdots & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N}{2}\right)!2^{N-1}} & 0 \\ 0 & 0 & \frac{1}{0!2!2^2} & \cdots & 0 & \frac{(-1)^{\frac{N-2}{2}}}{\left(\frac{N-2}{2}\right)!\left(\frac{N+2}{2}\right)!2^{N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{0!(N-1)!2^{N-1}} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{0!N!2^N} \end{bmatrix}_{(N+1)\times(N+1)}.$$

The approximate solutions $y_j(t)$, $(j = 1, 2, \ldots, k)$ given in relation (3) can be written in the matrix form

$$[y_j(t)] = \mathbf{J}(t)\mathbf{A}_j; \quad \mathbf{A}_j = \begin{bmatrix} a_{j,0} & a_{j,1} & \cdots & a_{j,N} \end{bmatrix}^T, \ j = 1, 2, \ldots, k.$$

By placing Eq. (4) into the above equation, we have the matrix relation

$$[y_j(t)] = \mathbf{T}(t)\mathbf{D}^T\mathbf{A}_j, \quad j = 1, 2, \ldots, k. \tag{5}$$

Substituting $q_r t$ instead of t in the expression (5), we gain the matrix form

$$[y_j(q_r t)] = \mathbf{T}(q_r t)\mathbf{D}^T\mathbf{A}_j, \quad j = 1, 2, \ldots, k. \tag{6}$$

The relation between the matrices $\mathbf{T}(q_r t)$ and $\mathbf{T}(t)$ is

$$\mathbf{T}(q_r t) = \mathbf{T}(t)\mathbf{B}(q_r), \tag{7}$$

so that

$$\mathbf{B}(q_r) = \begin{bmatrix} (q_r)^0 & 0 & \cdots & 0 \\ 0 & (q_r)^1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (q_r)^N \end{bmatrix}.$$

Substituting Eq. (7) into Eq. (6), we have matrix relation

$$[y_j(q_r t)] = \mathbf{T}(t)\mathbf{B}(q_r)\mathbf{D}^T\mathbf{A}_j. \tag{8}$$

The relation between the matrix $\mathbf{T}(t)$ and its derivative $\mathbf{T}^{(1)}(t)$ is

$$\mathbf{T}^{(1)}(t) = \mathbf{T}(t)\mathbf{B}^T, \tag{9}$$

where

$$\mathbf{B}^T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By using the relations (5) and (9), we get the matrix relations

$$\begin{aligned} y_j^{(1)}(t) &= \mathbf{T}^{(1)}(t)\mathbf{D}^T\mathbf{A}_j \\ &= \mathbf{T}(t)\mathbf{B}^T\mathbf{D}^T\mathbf{A}_j, \quad j = 1, 2, \ldots, k. \end{aligned} \tag{10}$$

Thus, the matrices $\mathbf{y}^{(0)}(t)$, $\mathbf{y}^{(1)}(t)$ and $\mathbf{y}^{(0)}(q_r t)$ can be expressed as follows:

$$\mathbf{y}^{(0)}(t) = \bar{\mathbf{T}}(t)\bar{\mathbf{D}}\mathbf{A}, \tag{11}$$

$$\mathbf{y}^{(1)}(t) = \bar{\mathbf{T}}(t)\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A} \tag{12}$$

and

$$\mathbf{y}^{(0)}(q_r t) = \bar{\mathbf{T}}(t)\bar{\mathbf{B}}_{q_r}\bar{\mathbf{D}}\mathbf{A}, \tag{13}$$

where

$$\mathbf{y}^{(0)}(t) = \begin{bmatrix} y_1^{(0)}(t) \\ y_2^{(0)}(t) \\ \vdots \\ y_k^{(0)}(t) \end{bmatrix}, \quad \mathbf{y}^{(1)}(t) = \begin{bmatrix} y_1^{(1)}(t) \\ y_2^{(1)}(t) \\ \vdots \\ y_k^{(1)}(t) \end{bmatrix}, \quad \mathbf{y}^{(0)}(q_r t) = \begin{bmatrix} y_1(q_r t) \\ y_2(q_r t) \\ \vdots \\ y_k(q_r t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_k \end{bmatrix},$$

$$\bar{\mathbf{T}}(t) = \begin{bmatrix} \mathbf{T}(t) & 0 & \cdots & 0 \\ 0 & \mathbf{T}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{T}(t) \end{bmatrix}_{k\times k}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^T & 0 & \cdots & 0 \\ 0 & \mathbf{B}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}^T \end{bmatrix}_{k\times k},$$

$$\bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D}^T & 0 & \cdots & 0 \\ 0 & \mathbf{D}^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{D}^T \end{bmatrix}_{k\times k} \quad \text{and} \quad \bar{\mathbf{B}}_{q_r} = \begin{bmatrix} \mathbf{B}(q_r) & 0 & \cdots & 0 \\ 0 & \mathbf{B}(q_r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B}(q_r) \end{bmatrix}_{k\times k}.$$

4. Method for solution

We can express the system (1) in the matrix form

$$\boldsymbol{\beta}(t)\mathbf{y}^{(1)}(t)=\boldsymbol{\gamma}(t)\mathbf{y}^{(0)}(t)+\sum_{r=0}^{R}\boldsymbol{\mu}_r(t)\mathbf{y}^{(0)}(q_r t)+\mathbf{g}(t), \tag{14}$$

where

$$\boldsymbol{\beta}(t)=\begin{bmatrix}\beta_{1,1}(t) & \beta_{1,2}(t) & \cdots & \beta_{1,k}(t)\\ \beta_{2,1}(t) & \beta_{2,2}(t) & \cdots & \beta_{2,k}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \beta_{k,1}(t) & \beta_{k,2}(t) & \cdots & \beta_{k,k}(t)\end{bmatrix},\qquad \mathbf{y}^{(1)}(t)=\begin{bmatrix}y_1^{(1)}(t)\\ y_2^{(1)}(t)\\ \vdots\\ y_k^{(1)}(t)\end{bmatrix},$$

$$\boldsymbol{\gamma}(t)=\begin{bmatrix}\gamma_{1,1}(t) & \gamma_{1,2}(t) & \cdots & \gamma_{1,k}(t)\\ \gamma_{2,1}(t) & \gamma_{2,2}(t) & \cdots & \gamma_{2,k}(t)\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{k,1}(t) & \gamma_{k,2}(t) & \cdots & \gamma_{k,k}(t)\end{bmatrix},$$

$$\mathbf{y}^{(0)}(t)=\begin{bmatrix}y_1^{(0)}(t)\\ y_2^{(0)}(t)\\ \vdots\\ y_k^{(0)}(t)\end{bmatrix},\qquad \boldsymbol{\mu}_r(t)=\begin{bmatrix}\mu_{1,1}^r(t) & \mu_{1,2}^r(t) & \cdots & \mu_{1,k}^r(t)\\ \mu_{2,1}^r(t) & \mu_{2,2}^r(t) & \cdots & \mu_{2,k}^r(t)\\ \vdots & \vdots & \ddots & \vdots\\ \mu_{k,1}^r(t) & \mu_{k,2}^r(t) & \cdots & \mu_{k,k}^r(t)\end{bmatrix},\qquad \mathbf{y}^{(0)}(q_r t)=\begin{bmatrix}y_1(q_r t)\\ y_2(q_r t)\\ \vdots\\ y_k(q_r t)\end{bmatrix}$$

$$\text{and}\quad \mathbf{g}(t)=\begin{bmatrix}g_1(t)\\ g_2(t)\\ \vdots\\ g_k(t)\end{bmatrix}.$$

To compute the unknown Bessel coefficients, we use the collocation points defined by

$$t_s=a+\frac{b-a}{N}s,\quad s=0,1,\ldots,N. \tag{15}$$

Now, we put the collocation points into Eq. (14) and thus we gain the matrix equation system

$$\boldsymbol{\beta}(t_s)\mathbf{y}^{(1)}(t_s)=\boldsymbol{\gamma}(t_s)\mathbf{y}^{(0)}(t_s)+\sum_{r=0}^{R}\boldsymbol{\mu}_r(t_s)\mathbf{y}^{(0)}(q_r t_s)+\mathbf{g}(t_s),\quad s=0,1,\ldots,N.$$

Briefly, this system can be written in the matrix form

$$\boldsymbol{\beta}\mathbf{Y}^{(1)}=\boldsymbol{\gamma}\mathbf{Y}^{(0)}+\sum_{r=0}^{R}\boldsymbol{\mu}_r\mathbf{Y}_{q_r}^{(0)}+\mathbf{G}, \tag{16}$$

where

$$\boldsymbol{\beta}=\begin{bmatrix}\boldsymbol{\beta}(t_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{\beta}(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\beta}(t_N)\end{bmatrix},\qquad \boldsymbol{\gamma}=\begin{bmatrix}\boldsymbol{\gamma}(t_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{\gamma}(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\gamma}(t_N)\end{bmatrix},\qquad \mathbf{Y}^{(0)}=\begin{bmatrix}\mathbf{y}^{(0)}(t_0)\\ \mathbf{y}^{(0)}(t_1)\\ \vdots\\ \mathbf{y}^{(0)}(t_N)\end{bmatrix},$$

$$\mathbf{Y}^{(1)}=\begin{bmatrix}\mathbf{y}^{(1)}(t_0)\\ \mathbf{y}^{(1)}(t_1)\\ \vdots\\ \mathbf{y}^{(1)}(t_N)\end{bmatrix},\qquad \boldsymbol{\mu}_r=\begin{bmatrix}\boldsymbol{\mu}_r(t_0) & 0 & \cdots & 0\\ 0 & \boldsymbol{\mu}_r(t_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \boldsymbol{\mu}_r(t_N)\end{bmatrix},\qquad \mathbf{Y}_{q_r}^{(0)}=\begin{bmatrix}\mathbf{y}^{(0)}(q_r t_0)\\ \mathbf{y}^{(0)}(q_r t_1)\\ \vdots\\ \mathbf{y}^{(0)}(q_r t_N)\end{bmatrix}\quad\text{and}\quad \mathbf{G}=\begin{bmatrix}\mathbf{g}(t_0)\\ \mathbf{g}(t_1)\\ \vdots\\ \mathbf{g}(t_N)\end{bmatrix}.$$

By using the relations (11)–(13) and the collocation points (15), we get

$$\mathbf{y}^{(0)}(t_s) = \bar{\mathbf{T}}(t_s)\bar{\mathbf{D}}\mathbf{A}, \quad s = 0, 1, \dots, N,$$
$$\mathbf{y}^{(1)}(t_s) = \bar{\mathbf{T}}(t_s)\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A}, \quad s = 0, 1, \dots, N,$$

and

$$\mathbf{y}^{(0)}(q_r t_s) = \bar{\mathbf{T}}(t_s)\bar{\mathbf{B}}_{q_r}\bar{\mathbf{D}}\mathbf{A}, \quad s = 0, 1, \dots, N,$$

or briefly, these systems can be expressed respectively as follows

$$\mathbf{Y}^{(0)} = \mathbf{T}\bar{\mathbf{D}}\mathbf{A}, \tag{17}$$
$$\mathbf{Y}^{(1)} = \mathbf{T}\bar{\mathbf{B}}\bar{\mathbf{D}}\mathbf{A} \tag{18}$$

and

$$\mathbf{Y}^{(0)}_{q_r} = \mathbf{T}\bar{\mathbf{B}}_{q_r}\bar{\mathbf{D}}\mathbf{A}, \tag{19}$$

where

$$\mathbf{T} = \begin{bmatrix} \bar{\mathbf{T}}(t_0) \\ \bar{\mathbf{T}}(t_1) \\ \vdots \\ \bar{\mathbf{T}}(t_N) \end{bmatrix}, \qquad \bar{\mathbf{X}}(t_s) = \begin{bmatrix} \mathbf{X}(t_s) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(t_s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(t_s) \end{bmatrix}_{k\times k}, \quad s = 0, 1, \dots, N.$$

When the relations (17)–(19) are substituted into Eq. (16), the fundamental matrix equation is obtained as

$$\left\{ \left(\boldsymbol{\beta}\mathbf{T}\bar{\mathbf{B}} - \boldsymbol{\gamma}\mathbf{T} - \sum_{r=0}^{R} \boldsymbol{\mu}_r \mathbf{T}\bar{\mathbf{B}}_{q_r} \right) \bar{\mathbf{D}} \right\} \mathbf{A} = \mathbf{G}. \tag{20}$$

We note the full dimension of the matrices $\boldsymbol{\beta}$, $\mathbf{T}$, $\bar{\mathbf{B}}$, $\bar{\mathbf{D}}$ and $\bar{\mathbf{B}}_{q_r}$ is $k(N+1) \times k(N+1)$ and also the matrices $\mathbf{A}$ and $\mathbf{G}$ have full dimension $k(N+1) \times 1$.

Now, let us write briefly the fundamental matrix equation (20) corresponding to Eq. (1) as

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \text{or} \quad [\mathbf{W}; \mathbf{G}]. \tag{21}$$

This system corresponds to a linear system of $k(N+1)$ algebraic equations in $k(N+1)$ unknown Bessel coefficients such that

$$\mathbf{W} = \left(\boldsymbol{\beta}\mathbf{T}\bar{\mathbf{B}} - \boldsymbol{\gamma}\mathbf{T} - \sum_{r=0}^{R} \boldsymbol{\mu}_r \mathbf{T}\bar{\mathbf{B}}_{q_r} \right) \bar{\mathbf{D}} = [w_{p,q}], \quad p, q = 1, 2, \dots, k(N+1).$$

With the aid of the relations (5) and (11), we obtain the corresponding matrix form to the conditions (2) as

$$[\bar{\boldsymbol{\phi}}\bar{\mathbf{T}}(a) + \bar{\boldsymbol{\psi}}\bar{\mathbf{T}}(b)]\bar{\mathbf{D}}\mathbf{A} = \boldsymbol{\lambda}, \tag{22}$$

where

$$\bar{\boldsymbol{\phi}} = \begin{bmatrix} \boldsymbol{\phi}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{\phi}_2 & \vdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\phi}_k \end{bmatrix}, \qquad \boldsymbol{\phi}_i = \begin{bmatrix} \phi_{i,1} \\ \phi_{i,2} \\ \vdots \\ \phi_{i,k} \end{bmatrix}, \qquad \bar{\boldsymbol{\psi}} = \begin{bmatrix} \boldsymbol{\psi}_1 & 0 & \cdots & 0 \\ 0 & \boldsymbol{\psi}_2 & \vdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\psi}_k \end{bmatrix},$$

$$\boldsymbol{\psi}_i = \begin{bmatrix} \psi_{i,1} \\ \psi_{i,2} \\ \vdots \\ \psi_{i,k} \end{bmatrix}, \qquad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{bmatrix}_{k\times 1}$$

and $i = 1, 2, \dots, k$.

Briefly, the matrix form (22) can be expressed as

$$\mathbf{UA} = \boldsymbol{\lambda} \quad \text{or} \quad [\mathbf{U}; \boldsymbol{\lambda}] \tag{23}$$

such that

$$\mathbf{U} = \left[\bar{\boldsymbol{\phi}}\bar{\mathbf{T}}(a) + \bar{\boldsymbol{\psi}}\bar{\mathbf{T}}(b)\right]\bar{\mathbf{D}}.$$

Finally, we replace the rows of the matrices $\mathbf{U}$ and $\boldsymbol{\lambda}$ by the rows of the matrices $\mathbf{W}$ and $\mathbf{G}$, respectively, and we have the new augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A} = \tilde{\mathbf{G}}. \tag{24}$$

For convenience, if the last k rows of the matrix $\mathbf{W}$ are replaced, the augmented matrix of the system (24) becomes as follows [3,7,9,11,13–16]:

$$[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,k(N+1)} & ; & g_1(t_0) \\ w_{2,1} & w_{2,2} & \cdots & w_{2,k(N+1)} & ; & g_2(t_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{k,1} & w_{k,2} & \cdots & w_{k,k(N+1)} & ; & g_k(t_0) \\ w_{k+1,1} & w_{k+1,2} & \cdots & w_{k+1,k(N+1)} & ; & g_1(t_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{kN,1} & w_{kN,2} & \cdots & w_{kN,k(N+1)} & ; & g_k(t_{N-1}) \\ v_{1,1} & v_{1,2} & \cdots & v_{1,k(N+1)} & ; & g_1(t_N) \\ v_{2,1} & v_{2,2} & \cdots & v_{2,k(N+1)} & ; & g_2(t_N) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k,1} & v_{k,2} & \cdots & v_{k,k(N+1)} & ; & g_k(t_N) \end{bmatrix}. \tag{25}$$

However, we do not have to replace the last rows. For example, if the matrix $\mathbf{W}$ is singular, then the rows that have the same factor or all zero are replaced.

If rank $\tilde{\mathbf{W}} = \text{rank}[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] = k(N+1)$, then we can write

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}. \tag{26}$$

The unknown the Bessel coefficients matrix $\mathbf{A}$ is determined solving this linear system and $a_{i,0}, a_{i,1}, \ldots, a_{i,N}$, $(i = 1, 2, \ldots, k)$ are substituted in Eq. (3). Thus we get the Bessel polynomial solutions

$$y_{i,N}(t) = \sum_{n=0}^{N} a_{i,n} J_n(t), \quad i = 1, 2, \ldots, k.$$

On the other hand, when $|\tilde{\mathbf{W}}| = 0$, if rank $\tilde{\mathbf{W}} = \text{rank}[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] < k(N+1)$, then we may find a particular solution. Otherwise if rank $\tilde{\mathbf{W}} \neq \text{rank}[\tilde{\mathbf{W}}; \tilde{\mathbf{G}}] < k(N+1)$, then it is not a solution.

5. Error estimation

In this section, we will given an error estimation for the Bessel approximate solution of the multi-pantograph equation system. This error estimation [20] has been presented by Shahmorad for integro-differential equations. We modify the error estimation studied in [20] for the multi-pantograph equation system.

Let us call $e_{i,N}(t) = y_i(t) - y_{i,N}(t)$ as the error function of the Bessel approximation $y_{i,N}(t)$ to $y_i(t)$, where $y_i(t)$ is the exact solution of (1) and (2). Hence, $y_{i,N}(t)$ satisfies the following problem:

$$\sum_{j=1}^{k} \beta_{i,j}(t) y_{j,N}^{(1)}(t) - \sum_{j=1}^{k} \gamma_{i,j}(t) y_{j,N}(t) - \sum_{r=1}^{R}\sum_{j=1}^{k} \mu_{i,j}^{r}(t) y_{j,N}(q_r t) = g_i(t) + R_{i,N}(t), \quad t \in [a, b] \tag{27}$$

$$\sum_{n=1}^{k} (\phi_{n,i} y_{n,N}(a) + \psi_{n,i} y_{n,N}(b)) = \lambda_i, \quad i = 1, 2, \ldots, k, \tag{28}$$

can be obtained by substituting $y_{j,N}(t)$ into Eq. (1)and here $R_{i,N}(t)$, $(i = 1, 2, \ldots, k)$ are the residual functions associated with $y_{j,N}(t)$.

We proceed to find an approximation $e_{i,N,M}(t)$ to $e_{i,N}(t)$ in the same way as we did before for the problem (1)–(2).

Subtracting (27) and (28) from (1) and (2), respectively, the error functions $e_{i,N}(t)$, $(i = 1, 2, \ldots, k)$ satisfy the equations

$$\sum_{j=1}^{k}\beta_{i,j}(t)e_{j,N}^{(1)}(t)-\sum_{j=1}^{k}\gamma_{i,j}(t)e_{j,N}(t)-\sum_{r=1}^{R}\sum_{j=1}^{k}\mu_{i,j}^{r}(t)e_{j,N}(q_r t)=-R_{i,N}(t),\quad t\in[a,b] \tag{29}$$

with the homogeneous conditions

$$\sum_{n=1}^{k}(\phi_{n,i}e_{n,N}(a)+\psi_{n,i}e_{n,N}(b))=0,\quad i=1,2,\ldots,k. \tag{30}$$

Solving this error problem in the same way as Section 2, we get the approximation $e_{i,N,M}(t)$. We note that in order to construct the Bessel approximation $e_{i,N,M}(t)$ to $e_{i,N}(t)$, the truncated limited M should be chosen such that $M \geq N$. Also, the right-hand side of (20) needs to be recomputed to $-R_{i,N}(t)$.

6. Numerical examples

In this section, we will give some numerical examples to illustrate the accuracy and effectiveness properties of the method and all of them were performed on a computer using a program written in MATLAB. In tables and figures, $y_i(t), y_{i,N}(t), |e_{i,N}(t)| = |y_i(t)-y_{i,N}(t)|$ and $|e_{i,N,M}(t)|$ for $i = 1, 2, \ldots, k$ and various values of N show respectively the exact solutions, approximate solutions, the absolute error functions and the estimated absolute error functions at the selected points of the given interval. Also, we compute the maximum absolute errors for Examples 1 and 3 by using

$$e_{i,N}=\|y_{i,N}(t)-y_i(t)\|_\infty=\max\{|y_{i,N}(t)-y_i(t)|,\ a\le t\le b\}.$$

Example 1. Let us first consider the pantograph equation system

$$\begin{cases} y_1^{(1)}(t)+2ty_2^{(1)}(t)=y_1(t)+ty_2(t)-y_1(0.5t)+t^2y_2(0.8t)+g_1(t)\\ ty_1^{(1)}(t)-y_2^{(1)}(t)=-ty_1(t)+3ty_2(t)-ty_1(0.3t)+y_2(2t)+g_2(t)\end{cases},\quad 0\le t\le 1, \tag{31}$$

with the initial conditions $y_1(0) = 1, y_2(0) = 1$ and the exact solution $y_1(t) = e^{-t}, y_2(t) = e^{t}$. Here, $g_1(t) = -2e^{-t} + te^{t} + e^{-\frac{1}{2}t} - t^2e^{\frac{4}{5}t}$, $g_2(t) = -4e^{-t} + te^{-\frac{3}{10}t} - e^{2t}$, $k = 2, q_1 = 0.5, q_2 = 0.8, q_3 = 0.3, q_4 = 2, \beta_{1,1}(t) = 1, \beta_{1,2}(t) = 2t, \beta_{2,1}(t) = t, \beta_{2,2}(t) = -1, \gamma_{1,1}(t) = 1, \gamma_{1,2}(t) = t, \gamma_{2,1}(t) = -t, \gamma_{2,2}(t) = 3, \mu^1_{1,1}(t) = -1, \mu^1_{1,2}(t) = 0, \mu^1_{2,1}(t) = 0, \mu^1_{2,2}(t) = 0, \mu^2_{1,1}(t) = 0, \mu^2_{1,2}(t) = t^2, \mu^2_{2,1}(t) = \mu^2_{2,2}(t) = 0, \mu^3_{1,1}(t) = 0, \mu^3_{1,2}(t) = 0, \mu^3_{2,1}(t) = -t, \mu^3_{2,2}(t) = 0, \mu^4_{1,1}(t) = \mu^4_{1,2}(t) = 0, \mu^4_{2,1}(t) = 0$ and $\mu^4_{2,2}(t) = 1$.

Let us seek the approximate the solutions

$$y_i(t)=\sum_{n=0}^{2}a_{i,n}J_n(t),\quad i=1,2$$

by the Bessel functions of first kind for $N = 2$. Firstly, the set of collocation points (15) for $N = 2$ is calculated as

$$\left\{t_0=0,\ t_1=\frac{1}{2},\ t_2=1\right\}$$

and from Eq. (20), we write the fundamental matrix equation of the problem as

$$\left\{\left(\boldsymbol{\beta}\mathbf{T}\bar{\mathbf{B}}-\boldsymbol{\gamma}\mathbf{T}-\boldsymbol{\mu}_1\mathbf{T}\bar{\mathbf{B}}_{0.5}-\boldsymbol{\mu}_2\mathbf{T}\bar{\mathbf{B}}_{0.8}-\boldsymbol{\mu}_3\mathbf{T}\bar{\mathbf{B}}_{0.3}-\boldsymbol{\mu}_4\mathbf{T}\bar{\mathbf{B}}_2\right)\bar{\mathbf{D}}\right\}\mathbf{A}=\mathbf{G}$$

where

$$\boldsymbol{\beta}=\begin{bmatrix}\boldsymbol{\beta}(0)&0&0\\0&\boldsymbol{\beta}(1/2)&0\\0&0&\boldsymbol{\beta}(1)\end{bmatrix},\quad \boldsymbol{\beta}(t)=\begin{bmatrix}1&2t\\t&-1\end{bmatrix},\quad \boldsymbol{\gamma}=\begin{bmatrix}\boldsymbol{\gamma}(0)&0&0\\0&\boldsymbol{\gamma}(1/2)&0\\0&0&\boldsymbol{\gamma}(1)\end{bmatrix},\quad \boldsymbol{\gamma}(t)=\begin{bmatrix}1&t\\-t&3\end{bmatrix},$$

$$\boldsymbol{\mu}_1=\begin{bmatrix}\boldsymbol{\mu}_1(0)&0&0\\0&\boldsymbol{\mu}_1(1/2)&0\\0&0&\boldsymbol{\mu}_1(1)\end{bmatrix},\quad \boldsymbol{\mu}_1(t)=\begin{bmatrix}-1&0\\0&0\end{bmatrix},\quad \boldsymbol{\mu}_2=\begin{bmatrix}\boldsymbol{\mu}_2(0)&0&0\\0&\boldsymbol{\mu}_2(1/2)&0\\0&0&\boldsymbol{\mu}_2(1)\end{bmatrix},$$

$$\boldsymbol{\mu}_2(t)=\begin{bmatrix}0&t^2\\0&0\end{bmatrix},\quad \boldsymbol{\mu}_3=\begin{bmatrix}\boldsymbol{\mu}_3(0)&0&0\\0&\boldsymbol{\mu}_3(1/2)&0\\0&0&\boldsymbol{\mu}_3(1)\end{bmatrix},\quad \boldsymbol{\mu}_3(t)=\begin{bmatrix}0&0\\-t&0\end{bmatrix},$$

$$\boldsymbol{\mu}_4=\begin{bmatrix}\boldsymbol{\mu}_4(0)&0&0\\0&\boldsymbol{\mu}_4(1/2)&0\\0&0&\boldsymbol{\mu}_4(1)\end{bmatrix},\quad \boldsymbol{\mu}_4(t)=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad \bar{\mathbf{B}}_{0.5}=\begin{bmatrix}\mathbf{B}(0.5)&0\\0&\mathbf{B}(0.5)\end{bmatrix},$$

$$\mathbf{B}(0.5)=\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1/4\end{bmatrix},\qquad \bar{\mathbf{B}}_{0.8}=\begin{bmatrix}\mathbf{B}(0.8)&0\\0&\mathbf{B}(0.8)\end{bmatrix},\qquad \mathbf{B}(0.8)=\begin{bmatrix}1&0&0\\0&4/5&0\\0&0&16/25\end{bmatrix},$$

$$\bar{\mathbf{B}}_{0.3}=\begin{bmatrix}\mathbf{B}(0.3)&0\\0&\mathbf{B}(0.3)\end{bmatrix},\qquad \mathbf{B}(0.3)=\begin{bmatrix}1&0&0\\0&3/10&0\\0&0&9/100\end{bmatrix},\qquad \bar{\mathbf{B}}_{2}=\begin{bmatrix}\mathbf{B}(2)&0\\0&\mathbf{B}(2)\end{bmatrix},$$

$$\mathbf{B}(2)=\begin{bmatrix}1&0&0\\0&2&0\\0&0&4\end{bmatrix},$$

$$\mathbf{T}=\begin{bmatrix}\bar{\mathbf{T}}(0)\\\bar{\mathbf{T}}(1/2)\\\bar{\mathbf{T}}(1)\end{bmatrix},\qquad \bar{\mathbf{T}}(0)=\begin{bmatrix}\mathbf{T}(0)&0\\0&\mathbf{T}(0)\end{bmatrix},\qquad \bar{\mathbf{T}}(1/2)=\begin{bmatrix}\mathbf{T}(1/2)&0\\0&\mathbf{T}(1/2)\end{bmatrix},$$

$$\bar{\mathbf{T}}(1)=\begin{bmatrix}\mathbf{T}(1)&0\\0&\mathbf{T}(1)\end{bmatrix},$$

$$\mathbf{T}(0)=\begin{bmatrix}1&0&0\end{bmatrix},\qquad \mathbf{T}(1/2)=\begin{bmatrix}1&1/2&1/4\end{bmatrix},\qquad \mathbf{T}(1)=\begin{bmatrix}1&1&1\end{bmatrix},$$

$$\bar{\mathbf{B}}=\begin{bmatrix}\mathbf{B}^T&0\\0&\mathbf{B}^T\end{bmatrix},\qquad \bar{\mathbf{D}}=\begin{bmatrix}\mathbf{D}^T&0\\0&\mathbf{D}^T\end{bmatrix},$$

$$\mathbf{B}^T=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix},\qquad \mathbf{D}^T=\begin{bmatrix}1&0&0\\0&1/2&0\\-1/4&0&1/8\end{bmatrix},\qquad \mathbf{G}=\begin{bmatrix}\mathbf{g}(0)\\\mathbf{g}(1/2)\\\mathbf{g}(1)\end{bmatrix},\qquad \mathbf{g}(0)=\begin{bmatrix}-1\\-5\end{bmatrix},$$

$$\mathbf{g}(1/2)=\begin{bmatrix}91/5308\\-1137/128\end{bmatrix},$$

$$\mathbf{g}(1)=\begin{bmatrix}534/1469\\-2050/117\end{bmatrix},\qquad \mathbf{A}=\begin{bmatrix}\mathbf{A}_1\\\mathbf{A}_2\end{bmatrix},\qquad \mathbf{A}_1=\begin{bmatrix}a_{1,0}\\a_{1,1}\\a_{1,2}\end{bmatrix}\quad\text{and}\quad \mathbf{A}_2=\begin{bmatrix}a_{2,0}\\a_{2,1}\\a_{2,2}\end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is

$$[\mathbf{W};\mathbf{G}]=\begin{bmatrix}0&1/2&0&0&0&0&;&-1\\0&0&0&-4&-12&0&;&-5\\-13/64&3/8&13/128&-767/800&13/40&167/1600&;&91/5308\\682/811&33/80&190/2389&-53/16&-7/4&-11/32&;&-1137/128\\-5/16&1/4&5/32&-259/100&1/10&59/200&;&534/1469\\491/400&23/20&309/800&-7/4&1&-9/8&;&-2050/117\end{bmatrix}.$$

The matrix form for initial conditions from Eq. (23) is computed as

$$[\mathbf{U};\lambda]=\begin{bmatrix}1&0&0&0&0&0&;&1\\0&0&0&1&0&0&;&1\end{bmatrix}.$$

Hence, the new augmented matrix based on conditions from system (25) is gained as

$$[\tilde{\mathbf{W}};\tilde{\mathbf{G}}]=\begin{bmatrix}0&1/2&0&0&0&0&;&-1\\0&0&0&-4&-12&0&;&-5\\-13/64&3/8&13/128&-767/800&13/40&167/1600&;&91/5308\\682/811&33/80&190/2389&-53/16&-7/4&-11/32&;&-1137/128\\1&0&0&0&0&0&;&1\\0&0&0&1&0&0&;&1\end{bmatrix}.$$

By solving this system, the unknown Bessel coefficients matrix is obtained as

$$\mathbf{A}=\begin{bmatrix}1&-2&1253/244&1&2&3527/486\end{bmatrix}^T.$$

As a result, substituting the elements of the column matrix $\mathbf{A}$ into Eq. (5) the approximate solutions for $N = 2$ are computed as

$$y_1(t)=1-t+0.391905863460t^2\quad\text{and}\quad y_2(t)=1+t+0.657150192245t^2.$$

By similar operations, the approximate solutions with our method of the problem for $N = 5, 8, 11$ are gained, respectively, as follows

$$
\begin{aligned}
y_{1,5}(t) &= 1 - t + 0.500436330298t^2 - 0.169071896590t^3 \\
&\quad + (0.453211032722e{-}1)t^4 - (0.5887045271424e{-}2)t^5, \\
y_{2,5}(t) &= 1 + t + 0.499376473859t^2 + 0.170699362647t^3 \\
&\quad + (0.330673898821e{-}1)t^4 + (0.151577614834e{-}1)t^5, \\
y_{1,8}(t) &= 1 - t + 0.499999692154t^2 - 0.166663227708t^3 + (0.416523590885e{-}1)t^4 \\
&\quad - (0.830704911691e{-}2)t^5 + (0.136900435052e{-}2)t^6 \\
&\quad - (0.193628078288e{-}3)t^7, +(0.221447878261e{-}4)t^8 \\
y_{2,8}(t) &= 1 + t + 0.500000609624t^2 + 0.166658955779t^3 + (0.417050096451e{-}1)t^4 \\
&\quad + (0.823866110202e{-}2)t^5 + (0.151291083330e{-}2)t^6 \\
&\quad + (0.112564650718e{-}3)t^7 + (0.531970346826e{-}4)t^8,
\end{aligned}
$$

and

$$
\begin{aligned}
y_{1,11}(t) &= 1 - t + 0.500000000025t^2 - 0.166666666934t^3 + (0.416666662402e{-}1)t^4 \\
&\quad - (0.833331468439e{-}2)t^5 + (0.138877859740e{-}2)t^6 - (0.198101749014e{-}3)t^7, \\
&\quad + (0.243164201063e{-}4)t^8 - (0.32417488128e{-}5)t^9 \\
&\quad + (0.666677452208e{-}7)t^{10} + (0.207089502818e{-}7)t^{11} \\
y_{2,11}(t) &= 1 + t + 0.499999999805t^2 + 0.166666670450t^3 + (0.416666356504e{-}1)t^4 \\
&\quad + (0.833347070321e{-}2)t^5 + (0.138852865099e{-}2)t^6 + (0.198998666854e{-}3)t^7 \\
&\quad + (0.241971829635e{-}4)t^8 + (0.314949007672e{-}5)t^9 \\
&\quad + (0.120599754901e{-}6)t^{10} + (0.572871185991e{-}7)t^{11}.
\end{aligned}
$$

To estimate the errors for $N = 3$, we consider the following error problem from (29)–(30)

$$
\begin{cases}
e^{(1)}_{1,3}(t) + 2te^{(1)}_{2,3}(t) - e_{1,3}(t) - te_{2,3}(t) + e_{1,3}(0.5t) - t^2e_{2,3}(0.8t) = -R_{1,3}(t) \\
te^{(1)}_{1,3}(t) - e^{(1)}_{2,3}(t) + te_{1,3}(t) - 3te_{2,3}(t) + te_{1,3}(0.3t) - e_{2,3}(2t) = -R_{2,3}(t) \\
e_{1,3}(0) = 0, \qquad e_{2,3}(0) = 0
\end{cases}, \quad 0 \le t \le 1, \tag{32}
$$

where the residual functions are

$$R_{1,3}(t) = y^{(1)}_{1,3}(t) + 2ty^{(1)}_{2,3}(t) - y_{1,3}(t) - ty_{2,3}(t) + y_{1,3}(0.5t) - t^2y_{2,3}(0.8t) - g_1(t)$$

and

$$R_{2,3}(t) = ty^{(1)}_{1,3}(t) - y^{(1)}_{2,3}(t) + ty_{1,3}(t) - 3ty_{2,3}(t) + ty_{1,3}(0.3t) - y_{2,3}(2t) - g_2(t).$$

By solving the error problem (32) for $M = 5$ with the method introduced in Section 2, the estimated error function approximations $e_{1,3,5}(t)$ and $e_{2,3,5}(t)$ are obtained as

$$
\begin{aligned}
e_{1,3,5}(t) &= 0.583204286515e{-}16 + (0.392594956117e{-}16)t + (0.432922666205e{-}2)t^2 \\
&\quad - (0.246566762199e{-}1)t^3 + (0.453211032723e{-}1)t^4 - (0.887045271426e{-}2)t^5
\end{aligned}
$$

and

$$
\begin{aligned}
e_{2,3,5}(t) &= 0.953313697749e{-}17 - (0.381325479100e{-}16)t + (0.307782638553e{-}1)t^2 \\
&\quad - (0.808411186122e{-}1)t^3 + (0.330673898821e{-}1)t^4 + (0.151577614834e{-}1)t^5.
\end{aligned}
$$

In Tables 1 and 2, we compare the actual and estimated absolute errors for $N = 3, 5, 8$ and $M = 5, 7, 8, 10, 13$. From these comparisons, we see that the estimated absolute errors are almost the same as with the actual absolute errors. Also, it is seen from Tables 1 and 2 that the estimated errors are closer to the actual errors while value M increases.

In Tables 3 and 4, we give the maximum absolute errors obtained by using $e_{i,N} = \|y_{i,N}(t) - y_i(t)\|_\infty = \max\{|y_{i,N}(t) - y_i(t)|,\ 0 \le t \le 1\}$ for $N = 2, 5, 8, 11$ and $i = 1, 2$.

Fig. 1 shows comparison of the absolute error functions obtained by the present method for $N = 2, 5, 8, 11$ of system (31). In Tables 3 and 4 and Fig. 1, we see that the errors decrease while value N increases.

Table 1
Comparison of actual and estimated absolute errors for $N = 3, 5, 8$ and $M = 5, 7, 8, 10, 13$ of $y_1(t)$ of the system (31).

t_i	Actual absolute errors	Estimated absolute errors	
	$\lvert e_{1,3}(t_i)\rvert = \lvert y_1(t_i) - y_{1,3}(t_i)\rvert$	$\lvert e_{1,3,5}(t_i)\rvert$	$\lvert e_{1,3,8}(t_i)\rvert$
0	0	5.8320e−017	1.5838e−018
0.2	4.1791e−005	4.5591e−005	4.1790e−005
0.4	1.8548e−004	1.8404e−004	1.8548e−004
0.6	1.4068e−003	1.4165e−003	1.4068e−003
0.8	5.7610e−003	5.8033e−003	5.7610e−003
1	1.6188e−002	1.6123e−002	1.6187e−002
	$\lvert e_{1,5}(t_i)\rvert = \lvert y_1(t_i) - y_{1,5}(t_i)\rvert$	$\lvert e_{1,5,7}(t_i)\rvert$	$\lvert e_{1,5,10}(t_i)\rvert$
0	0	1.0067e−017	3.5232e−017
0.2	3.8002e−006	3.7884e−006	3.8002e−006
0.4	1.4478e−006	1.4650e−006	1.4478e−006
0.6	9.7617e−006	9.7966e−006	9.7618e−006
0.8	4.2330e−005	4.2861e−005	4.2330e−005
1	6.4357e−005	6.4497e−005	6.4357e−005
	$\lvert e_{1,8}(t_i)\rvert = \lvert y_1(t_i) - y_{1,8}(t_i)\rvert$	$\lvert e_{1,8,10}(t_i)\rvert$	$\lvert e_{1,8,13}(t_i)\rvert$
0	0	1.5220e−018	2.6063e−018
0.2	5.0013e−010	4.9973e−010	5.0013e−010
0.4	9.4022e−010	9.4533e−010	9.4022e−010
0.6	9.3624e−009	9.3830e−009	9.3624e−009
0.8	3.5070e−009	3.7523e−009	3.5070e−009
1	1.4569e−007	1.4587e−007	1.4569e−007

Table 2
Comparison of actual and estimated absolute errors for $N = 3, 5, 8$ and $M = 5, 7, 8, 10, 13$ of $y_2(t)$ of the system (31).

t_i	Absolute errors	Estimated absolute errors	
	$\lvert e_{1,3}(t_i)\rvert = \lvert y_1(t_i) - y_{1,3}(t_i)\rvert$	$\lvert e_{1,3,5}(t_i)\rvert$	$\lvert e_{1,3,8}(t_i)\rvert$
0	0	9.5331e−018	1.4889e−017
0.2	6.4651e−004	6.4216e−004	6.4651e−004
0.4	7.5039e−004	7.5243e−004	7.5039e−004
0.6	9.0930e−004	9.1731e−004	9.0931e−004
0.8	3.1507e−003	3.1813e−003	3.1507e−003
1	1.8569e−003	1.8377e−003	1.8568e−003
	$\lvert e_{1,5}(t_i)\rvert = \lvert y_1(t_i) - y_{1,5}(t_i)\rvert$	$\lvert e_{1,5,7}(t_i)\rvert$	$\lvert e_{1,5,10}(t_i)\rvert$
0	0	3.4184e−018	8.8333e−019
0.2	4.3460e−006	4.3324e−006	4.3460e−006
0.4	2.0380e−006	2.0588e−006	2.0380e−006
0.6	8.0062e−006	8.0261e−006	8.0062e−006
0.8	3.0613e−005	3.0964e−005	3.0614e−005
1	1.9159e−005	1.9058e−005	1.9159e−005
	$\lvert e_{1,8}(t_i)\rvert = \lvert y_1(t_i) - y_{1,8}(t_i)\rvert$	$\lvert e_{1,8,10}(t_i)\rvert$	$\lvert e_{1,8,13}(t_i)\rvert$
0	0	1.2013e−019	4.5381e−019
0.2	6.6131e−010	6.5930e−010	6.6131e−010
0.4	1.3769e−009	1.3819e−009	1.3769e−009
0.6	7.9768e−009	7.9932e−009	7.9768e−009
0.8	4.5566e−009	4.7252e−009	4.5566e−009
1	8.0210e−008	8.0244e−008	8.0210e−008

Table 3
The maximum absolute error $e_{1,N}$ for $N = 2, 5, 8, 11$ of $y_1(x)$ of system (31).

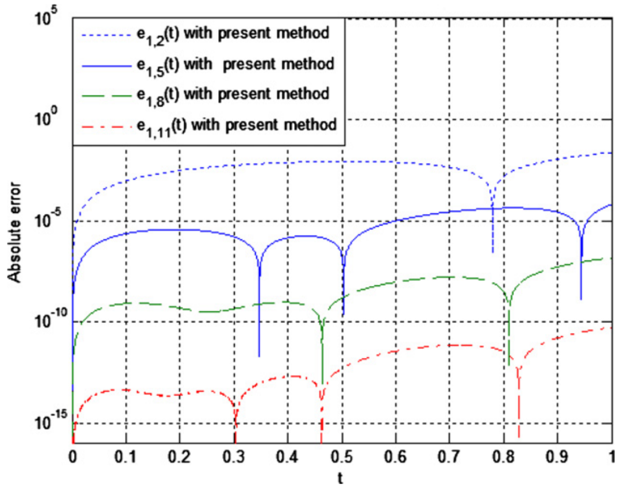
N	2	5	8	11
$e_{1,N}$	2.4026×10^{-2}	6.4357×10^{-5}	1.4569×10^{-7}	5.4326×10^{-11}

Example 2. As a second example, consider the following pantograph equation system

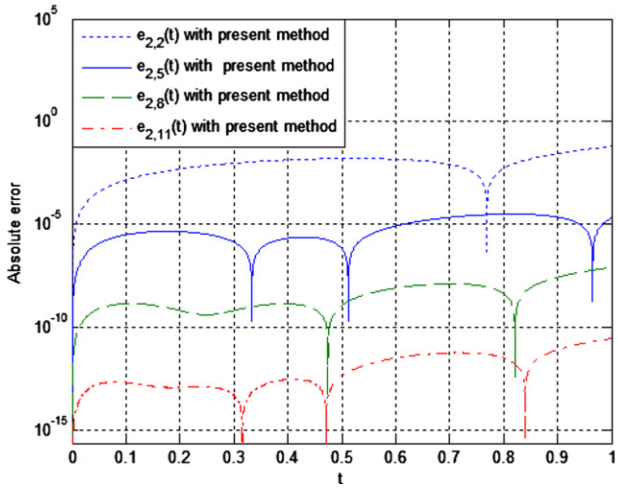
$$\begin{cases} y_1^{(1)}(t) = ty_1(t) + 2t^2y_2(t) - ty_1(2t) + 3y_2(3t) - 1 - 13t + 9t^2 - 7t^3 + 2t^4 \\ y_1^{(1)}(t) - ty_2^{(1)}(t) = 3ty_2(t) + t^2y_1(0.5t) - ty_2(3t) - 2 - t - t^2 - 5t^3 - \dfrac{1}{4}t^4 \end{cases}, \quad 0 \le t \le 1,$$

Table 4
The maximum absolute error $e_{2,N}$ for $N = 2, 5, 8, 11$ of $y_2(x)$ of system (31).

N	2	5	8	11
$e_{2,N}$	6.1132×10^{-2}	1.9159×10^{-5}	8.0210×10^{-8}	2.7323×10^{-11}



(a) Comparison of the absolute error functions $e_{1,N}(t)$ for $N = 2, 5, 8, 11$.



(b) Comparison of the absolute error functions $e_{2,N}(t)$ for $N = 2, 5, 8, 11$.

Fig. 1. For $N = 2, 5, 8, 11$ of the system (31), (a) Comparison of the absolute error functions $e_{1,N}(t)$ for $y_{1,N}(t)$ and (b) Comparison of the absolute error functions $e_{2,N}(t)$ for $y_{2,N}(t)$.

with the boundary conditions $y_1(0) = 3$ and $y_2(1) = 3$ such that $k = 2, q_1 = 2, q_2 = 3, q_3 = 0.5, \beta_{1,1}(t) = 1, \beta_{1,2}(t) = 0, \beta_{2,1}(t) = 1, \beta_{2,2}(t) = -t, \gamma_{1,1}(t) = t, \gamma_{1,2}(t) = 2t^2, \gamma_{2,1}(t) = 0, \gamma_{2,2}(t) = 3t, \mu^1_{1,1}(t) = -t, \mu^1_{1,2}(t) = 0, \mu^1_{2,1}(t) = 0, \mu^1_{2,2}(t) = 0, \mu^2_{1,1}(t) = 3, \mu^2_{1,2}(t) = 0, \mu^2_{2,1}(t) = 0, \mu^2_{2,2}(t) = -t, \mu^3_{1,1}(t) = 0, \mu^3_{1,2}(t) = 0, \mu^3_{2,1}(t) = t^2, \mu^3_{2,2}(t) = 0$.

From Eq. (20), the fundamental matrix equation of the problem is

$$\{(\boldsymbol{\beta}\mathbf{T}\bar{\mathbf{B}} - \boldsymbol{\gamma}\mathbf{T} - \boldsymbol{\mu}_1\mathbf{T}\bar{\mathbf{B}}_2 - \boldsymbol{\mu}_2\mathbf{T}\bar{\mathbf{B}}_3 - \boldsymbol{\mu}_3\mathbf{T}\bar{\mathbf{B}}_{0.5})\bar{\mathbf{D}}\}\mathbf{A} = \mathbf{G}.$$

By applying the present method for $N = 2$ as in Example 1, we find the approximate solutions of the problem for $N = 2$ as $y_1(t) = t^2 - 2t + 3$ and $y_2(t) = -t^2 + 5t - 1$ which are the exact solutions.

Table 5
The maximum absolute error $e_{1,N}$ for $N = 2, 5, 8, 11$ of $y_1(x)$ of the system (33).

N	5	8	11
$e_{1,N}$	1.2845×10^{-2}	5.5941×10^{-6}	3.4157×10^{-8}

Table 6
The maximum absolute error $e_{2,N}$ for $N = 2, 5, 8, 11$ of $y_2(x)$ of the system (33).

N	5	8	11
$e_{2,N}$	5.2365×10^{-3}	2.3075×10^{-6}	1.4039×10^{-8}

Table 7
The maximum absolute error $e_{3,N}$ for $N = 2, 5, 8, 11$ of $y_3(x)$ of the system (33).

N	5	8	11
$e_{2,N}$	1.3106×10^{-2}	5.4634×10^{-6}	3.3499×10^{-8}

Example 3. Finally, let us consider the system of pantograph equations given by

$$\begin{cases} ty_1^{(1)}(t) + y_2^{(1)}(t) - 2y_3^{(1)}(t) = y_2(t) - 2ty_3(t) + y_1(2t) - t^2y_2(3t) - y_3(3t) + ty_3(0.5t) + g_1(t) \\ y_1^{(1)}(t) - y_2^{(1)}(t) = ty_1(t) - y_3(t) + 3ty_1(0.5t) - y_2(0.5t) + ty_2(1.5t) + y_3(0.7t) + g_2(t) \\ y_2^{(1)}(t) - 2ty_3^{(1)}(t) = y_1(t) + 3y_2(t) - y_1(2t) + ty_2(0.8t) - y_3(0.8t) + g_3(t) \end{cases}, \quad 0 \le t \le 1, \tag{33}$$

with the initial conditions $y_1(0) = 0$, $y_2(0) = 1$, $y_3(0) = 1$ and the exact solutions $y_1(t) = \sin(t)$, $y_2(t) = \cos(t)$, $y_3(t) = e^t$. Here, $k = 3$, $q_1 = 2$, $q_2 = 3$, $q_3 = 0.5$, $q_4 = 1.5$, $q_5 = 0.7$, $q_6 = 0.8$, $\beta_{1,1}(t) = t$, $\beta_{1,2}(t) = 1$, $\beta_{1,3}(t) = -2$, $\beta_{2,1}(t) = 1$, $\beta_{2,2}(t) = -1$, $\beta_{2,3}(t) = 0$, $\beta_{3,1}(t) = 0$, $\beta_{3,2}(t) = 1$, $\beta_{3,3}(t) = 2t$, $\gamma_{1,1}(t) = 0$, $\gamma_{1,2}(t) = 1$, $\gamma_{1,3}(t) = -2t$, $\gamma_{2,1}(t) = t$, $\gamma_{2,2}(t) = 0$, $\gamma_{2,3}(t) = -1$, $\gamma_{3,1}(t) = 1$, $\gamma_{3,2}(t) = 3$, $\gamma_{3,3}(t) = 0$, $\mu^1_{1,1}(t) = 1$, $\mu^1_{1,2}(t) = 0$, $\mu^1_{1,3}(t) = 0$, $\mu^1_{2,1}(t) = 0$, $\mu^1_{2,2}(t) = 0$, $\mu^1_{2,3}(t) = 0$, $\mu^1_{3,1}(t) = -1$, $\mu^1_{3,2}(t) = 0$, $\mu^1_{3,3}(t) = 0$, $\mu^2_{1,1}(t) = 0$, $\mu^2_{1,2}(t) = -t^2$, $\mu^2_{1,3}(t) = -1$, $\mu^2_{2,1}(t) = \mu^2_{2,2}(t) = \mu^2_{2,3}(t) = 0$, $\mu^2_{3,1}(t) = \mu^2_{3,2}(t) = \mu^2_{3,3}(t) = 0$, $\mu^3_{1,1}(t) = 0$, $\mu^3_{1,2}(t) = 0$, $\mu^3_{1,3}(t) = t$, $\mu^3_{2,1}(t) = 3t$, $\mu^3_{2,2}(t) = -1$, $\mu^3_{2,3}(t) = 0$, $\mu^3_{3,1}(t) = 0$, $\mu^3_{3,2}(t) = 0$, $\mu^3_{3,3}(t) = 0$, $\mu^4_{1,1}(t) = 0$, $\mu^3_{1,2}(t) = 0$, $\mu^3_{1,3}(t) = 0$, $\mu^4_{2,1}(t) = 0$, $\mu^4_{2,2}(t) = t$, $\mu^4_{2,3}(t) = 0$, $\mu^4_{3,1}(t) = 0$, $\mu^4_{3,2}(t) = 0$, $\mu^4_{3,3}(t) = 0$, $\mu^5_{1,1}(t) = 0$, $\mu^5_{1,2}(t) = 0$, $\mu^5_{1,3}(t) = 0$, $\mu^5_{2,1}(t) = 0$, $\mu^5_{2,2}(t) = 0$, $\mu^5_{2,3}(t) = 1$, $\mu^5_{3,1}(t) = 0$, $\mu^5_{3,2}(t) = 0$, $\mu^5_{3,3}(t) = 0$, $\mu^6_{1,1}(t) = 0$, $\mu^6_{1,2}(t) = 0$, $\mu^6_{1,3}(t) = 0$, $\mu^6_{2,1}(t) = 0$, $\mu^6_{2,2}(t) = 0$, $\mu^6_{2,3}(t) = 1$, $\mu^6_{3,1}(t) = 0$, $\mu^6_{3,2}(t) = t$, $\mu^6_{3,3}(t) = -1$, $g_3(t) = -2\sin(t) + 2te^t - 3\cos(t) + \sin(2t) + (1-t)e^{\frac{4}{5}t}$,

$$g_1(t) = (t-1)\cos(t) - \sin(t) + (2t-2)e^t - \sin(2t) + t^2\cos(3t) + e^{3t} - te^{\frac{1}{2}t}$$

and

$$g_2(t) = \cos(t) + (1-t)\sin(t) + e^t - 3t\sin\left(\frac{1}{2}t\right) + \cos\left(\frac{1}{2}t\right) - t\cos\left(\frac{3}{2}t\right) - e^{\frac{7}{10}t}.$$

The fundamental matrix equation of Example 3 from Eq. (20) is written as

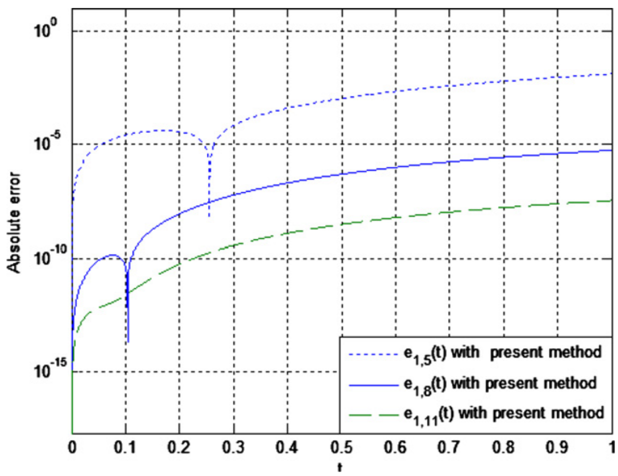
$$\{(\boldsymbol{\beta}\mathbf{T}\bar{\mathbf{B}} - \boldsymbol{\gamma}\mathbf{T} - \boldsymbol{\mu}_1\mathbf{T}\bar{\mathbf{B}}_2 - \boldsymbol{\mu}_2\mathbf{T}\bar{\mathbf{B}}_3 - \boldsymbol{\mu}_3\mathbf{T}\bar{\mathbf{B}}_{0.5} - \boldsymbol{\mu}_4\mathbf{T}\bar{\mathbf{B}}_{1.5} - \boldsymbol{\mu}_5\mathbf{T}\bar{\mathbf{B}}_{0.7} - \boldsymbol{\mu}_6\mathbf{T}\bar{\mathbf{B}}_{0.8})\bar{\mathbf{D}}\}\mathbf{A} = \mathbf{G}.$$

By following the procedure defined in Section 3, we obtain the approximate solutions for $N = 5, 8, 11$.

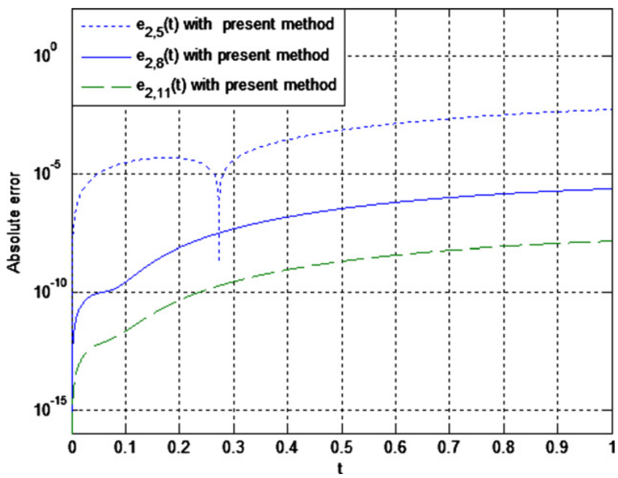
In Tables 5–7, we give the maximum absolute errors gained by the present method for $N = 5, 8, 11$ of the system (33). Fig. 2 displays the absolute error functions obtained by the present method for $N = 5, 8, 11$ of the system (33).

7. Conclusions

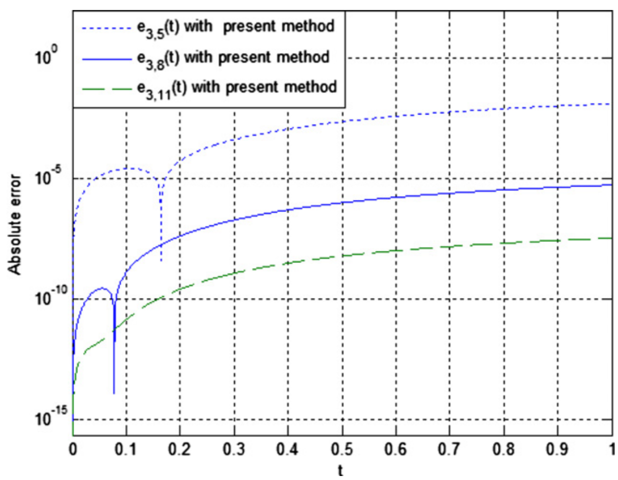
In this study, we present a numerical scheme based on the Bessel functions of the first kind for solving multi-pantograph equation systems. An interesting feature of this method is to find the analytical solutions if the system has exact solutions that are polynomial functions. This feature can be seen from Example 2. If the exact solutions of system are not polynomial functions, then a good approximation can be gained by using the proposed method. The errors can be estimated by using the error estimation given in Section 4. We have estimated the errors by Eqs. (29)–(30) in Example 1 and we see that the estimated absolute errors are almost the same as with the actual absolute errors. In addition, it is seen from tables and figures that the errors decrease as N increases. However, there may be big computational errors for large values of N in the process due to rounding errors in computing. So the results may not be accurate enough for large values of N such as ($N \gg 20$). The numerical results show that the present method is an accurate and reliable technique for pantograph equation systems.



(a) Comparison of the absolute error functions $e_{1,N}(t)$ for $N = 5, 8, 11$.



(b) Comparison of the absolute error functions $e_{2,N}(t)$ for $N = 5, 8, 11$.



(c) Comparison of the absolute error functions $e_{3,N}(t)$ for $N = 5, 8, 11$.

Fig. 2. For $N = 5, 8, 11$ of the system (33), (a) Comparison of the absolute error functions $e_{1,N}(t)$ for $y_{1,N}(t)$, (b) Comparison of the absolute error functions $e_{2,N}(t)$ for $y_{2,N}(t)$ and (c) Comparison of the absolute error functions $e_{3,N}(t)$ for $y_{3,N}(t)$.

Moreover, the approximate solutions can be very easily calculated using computer programs such as Matlab, Maple and Mathematica.

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